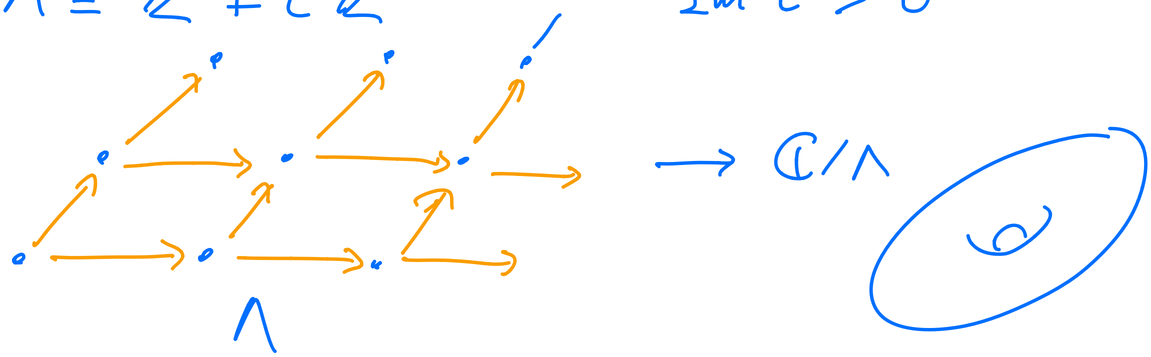
Holomorphic Dynamics – Lecture 7

Dynamics on the torus

$S = \mathbb{C}/\Lambda$

$\Lambda = \mathbb{Z} + \tau\mathbb{Z}$ $\qquad$ $\operatorname{Im}\tau > 0$



Thm Every holo $f : \mathbb{T} = \mathbb{C}/\Lambda \longrightarrow \mathbb{T}$ is affine; i.e. it is of form

$$f(z) = \alpha z + c \pmod{\Lambda}$$

with $\alpha \in \mathbb{C}$.

Hence $J(f)$ is <u>empty</u> if $|\alpha| \leq 1$ and $J(f) = \mathbb{T}$ if $|\alpha| > 1$.

Pf Consider universal cover:

$$\begin{tikzcd} \mathbb{C} \arrow[r, "F"] \arrow[d, "\pi"'] & \mathbb{C} \arrow[d, "\pi"] \\ \mathbb{C}/\Lambda \arrow[r, "f"] & \mathbb{C}/\Lambda \end{tikzcd} \qquad (*)$$

__Claim__: $\exists\, \lambda_1, \lambda_2 \in \Lambda$ s.t.

$$F(z+1) = F(z) + \lambda_1$$

$$F(z+\tau) = F(z) + \lambda_2$$

where $\Lambda = \mathbb{Z} + \tau\mathbb{Z}$

__Indeed__: $F(z+1) - F(z) \in \Lambda$ by commutativity of $\circledast$

And since $F(z+1) - F(z)$ is continuous and Λ is tot. disconnected, there is $\lambda_1 \in \Lambda$ s.t. $F(z+1) - F(z) = \lambda_1$

Consider $\boxed{g(z) := F(z) - \lambda_1 z}$

$$g(z+1) = F(z+1) - \lambda_1 z - \lambda_1$$

$$= F(z) + \cancel{\lambda_1} - \lambda_1 z - \cancel{\lambda_1} = g(z)$$

$$g(z+\tau) = F(z+\tau) - \lambda_1 z - \lambda_1 \tau$$

$$= F(z) + \lambda_2 - \lambda_1 z - \lambda_1 \tau =$$

$$= g(z) + (\lambda_2 - \lambda_1 \tau)$$

$$g : \mathbb{C}/\Lambda \longrightarrow \mathbb{C}/(\lambda_2 - \lambda_1\tau)\mathbb{Z} \cong \mathbb{C}\setminus\{0\} \subseteq \mathbb{C}$$

$$\begin{tikzcd} \mathbb{C} \arrow[d] \arrow[drr, dashed, bend left, "G"] & & \\ \mathbb{C}/\Lambda \arrow[r, "g"] & \mathbb{C}^* \arrow[r, hook] & \mathbb{C} \end{tikzcd}$$

The image $g(\mathbb{C}/\Lambda)$ is a compact subset of $\mathbb{C}$. So G is entire & bounded, hence constant.

So $g(z) = F(z) - \lambda_1 z = c$

$\Longrightarrow F(z) = \lambda_1 z + c$ <u>AFFINE</u>

Two Cases

1. $f : \mathbb{T} \to \mathbb{T}$, $f(z) = \alpha z + c$, $|\alpha| = 1$

Since f is defined on $\mathbb{C}/\Lambda$, $\alpha\Lambda \subset \Lambda$.

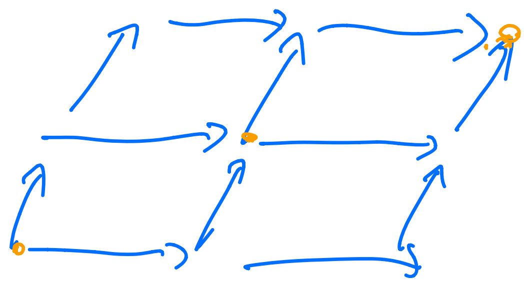
It turns out that α must be a root of 1 of period $2, 3, 4, 6$.

$f(z) = \alpha z$ $\qquad \alpha^p = 1$

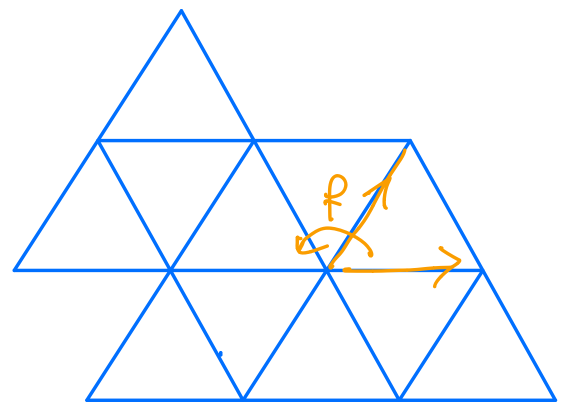
$\Longrightarrow \quad f^p(z) = z$

Examples $\quad f$ auto of $\mathbb{C}/\Lambda$

of order 2



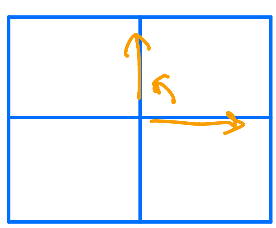
$f(z) = -z$



✓ order 3

✓ order 4

✓ order 6 $\qquad \Lambda$ has complex multiplication



$J(f) = \phi$

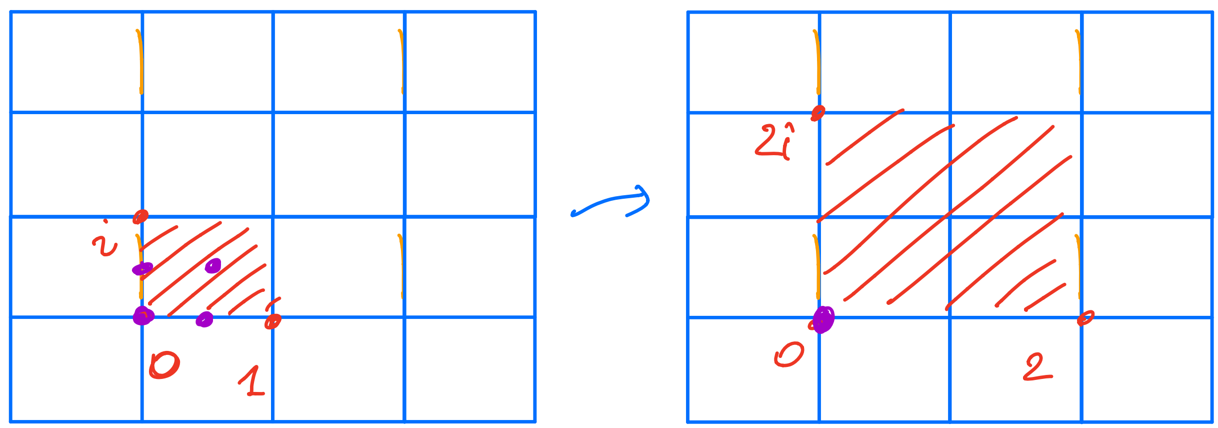
② $f(z) = \alpha z \qquad |\alpha| > 1$

$\Lambda = \mathbb{Z} + i\mathbb{Z}$

$f(z) = d z \pmod \Lambda$

well defined $\qquad \deg f = d^2$

$d=2$



$\deg f = 4$

$J(f) = \mathbb{T}$

Every periodic point is REPELLING

$f^p(z) = z \pmod{\Lambda}$

$d^p z + \lambda = z$

$(d^p - 1) z = -\lambda$

$z = \dfrac{-\lambda}{d^p - 1}$ $\qquad \Lambda = \mathbb{Z} + i\mathbb{Z}$

$n, m \in \mathbb{Z}$

$$z = \frac{n + im}{d^p - 1} = \frac{n}{d^p - 1} + i\,\frac{m}{d^p - 1}$$

Hence repelling periodic points are dense $\Longrightarrow$ $J(f) = \mathbb{T}$.

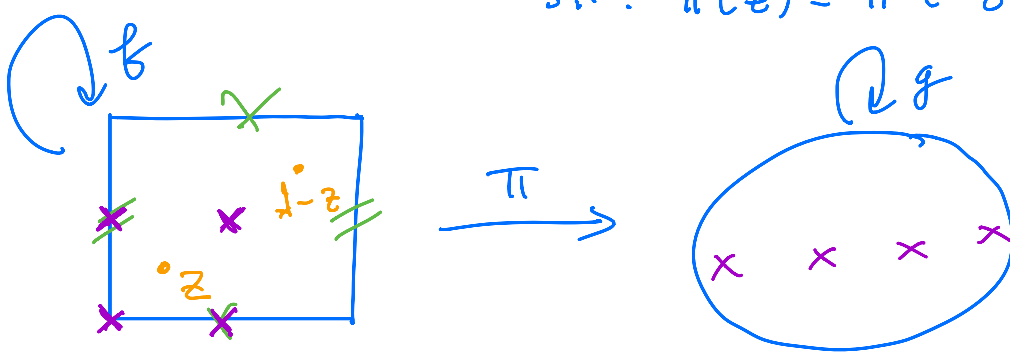
Thm The Julia set is the closure of the set of repelling periodic points.

Lattès maps

$f(z) = dz$ on $\mathbb{C}/\Lambda$

$\pi : \mathbb{C}/\Lambda \longrightarrow \hat{\mathbb{C}}$

s.t. $\pi(z) = \pi(-z)$

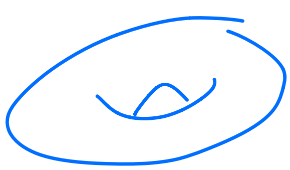


$-z$

hyperelliptic involution

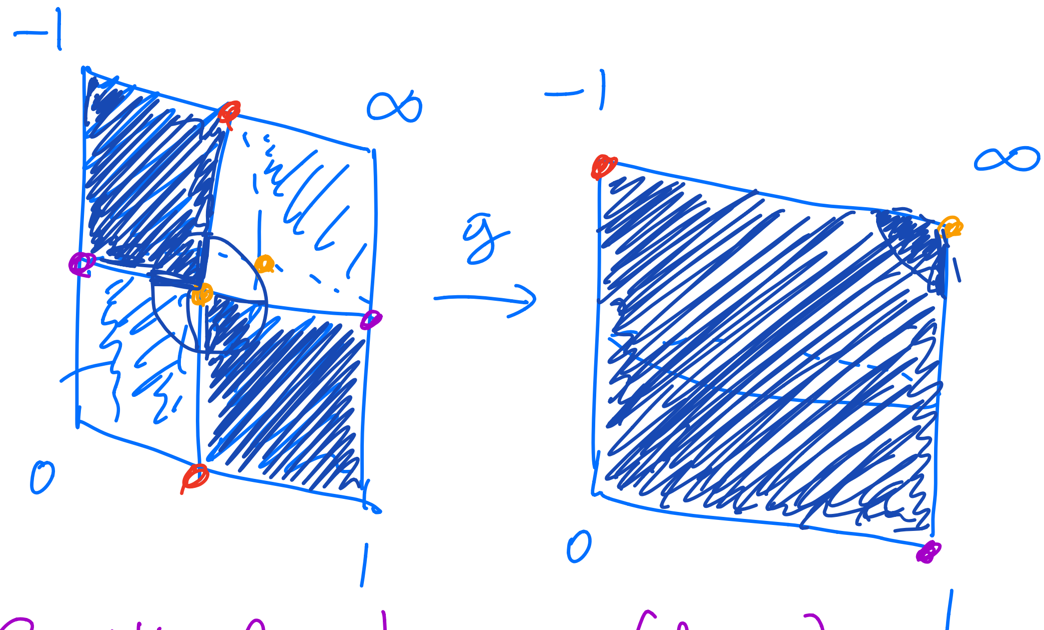
$\eta : \mathbb{C}/\Lambda \to \mathbb{C}/\Lambda$

$\eta(z) = -z$



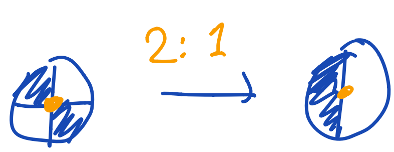
$$\begin{array}{ccc} \mathbb{C}/\Lambda & \xrightarrow[\pi]{2} & \mathbb{C} \\ d^2 \; f \downarrow & & \downarrow g \; d^2 \\ \mathbb{C}/\Lambda & \xrightarrow[\pi]{2} & \hat{\mathbb{C}} \end{array}$$

$\cancel{\deg g}$?



6 critical pts (d = 4)

\# critical points of rational map of degree d
$= 2d - 2$



Def.: A rational map is POSTCRITICALLY FINITE if the orbit of every critical point is finite

Exercise find formula for $g(z)$

Dynamics on Hyperbolic Surfaces

Thm If S is a hyperbolic R.S. and $f: S \to S$ a holo map, then $J(f) = \emptyset$. Moreover, either:

(a) every orbit converges towards a unique attracting fixed point

(b) every orbit diverges to infinity w.r.t. the Poincaré metric on S

(c) f is an automorphism of finite order

(d) f is conjugate to $f(z) = e^{2\pi i \alpha} z$ with $\alpha \in \mathbb{R} \setminus \mathbb{Q}$, and $S \simeq \mathbb{D}$ or $S \simeq \mathbb{D}^*$, or $S \simeq \{1 < |z| < r\}$.

Moreover, if U is hyperbolic open subset of $\mathbb{C}$ and $f: U \to U$

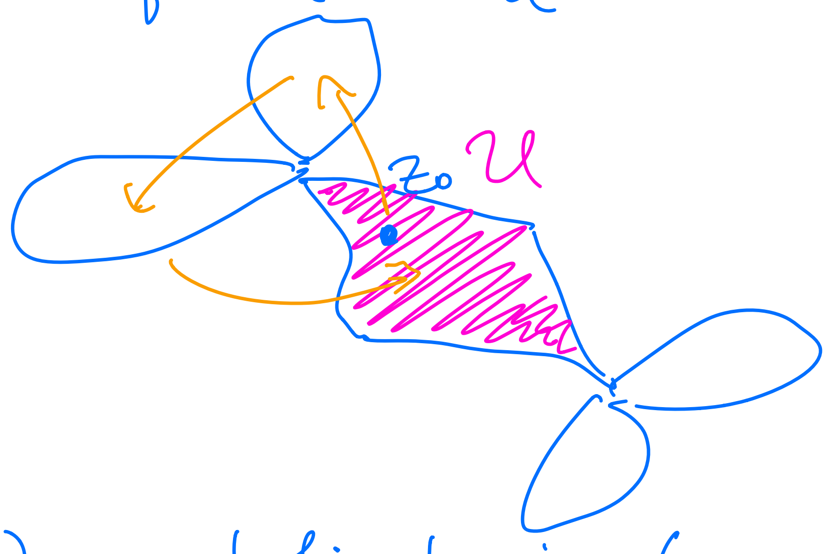
extends holomorphically throughout a neighborhood of $\overline{\mathcal{U}}$, then

$f^n(z) \longrightarrow \hat{z} \in \partial\mathcal{U}$ for all $z \in \mathcal{U}$.

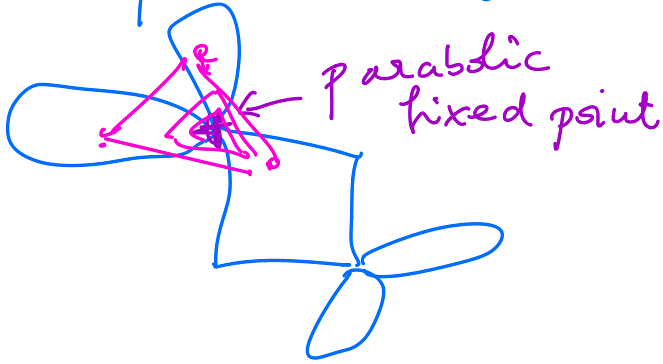
Examples

(a) $\mathcal{U}$ immediate attracting basin of a periodic attracting point z_0 of period p,

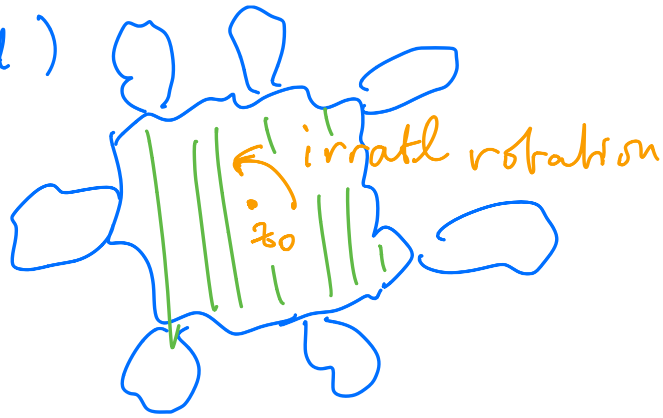
$$f^p : \mathcal{U} \longrightarrow \mathcal{U}$$



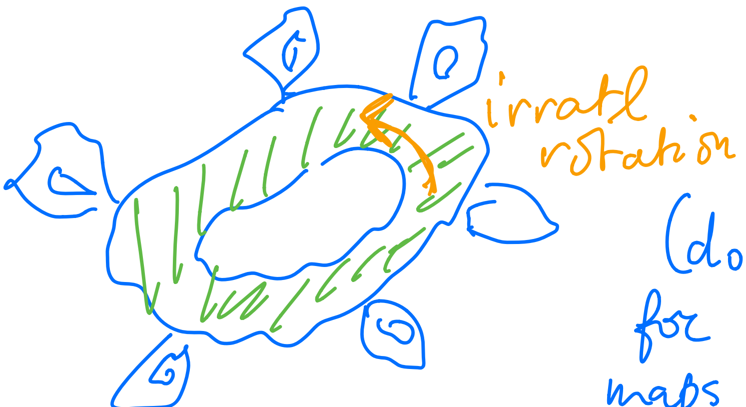
(b) parabolic basin (z_0 parabolic periodic point)



(d)



SIEGEL DISK



HERMAN RING

(do not appear for polynomial maps of $\mathbb{C}$)

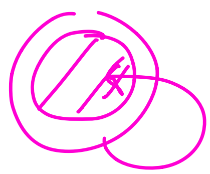
Thm (Denjoy-Wolff)

A holo $f: \mathbb{D} \to \mathbb{D}$ is either a rotation about $z_0 \in \mathbb{D}$, or the iterates (f^n) converge, unif. on compact subsets of $\mathbb{D}$, to a constant map $f(z) = c \in \overline{\mathbb{D}}$.

Pf Suppose there is no fixed point z_0 for f.

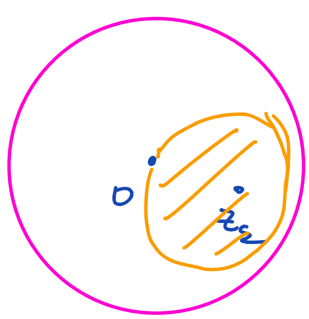
[If there is z_0, $f(z_0) = z_0$, then it is either a rotation or it contracts the hyperbolic metric, Then z_0 should be attracting, hence $f^n \to \{z_0\}$].

Consider $f_\varepsilon(z) := (1-\varepsilon) f(z)$



then there is z_ε s.t. $f_\varepsilon(z_\varepsilon) = z_\varepsilon$,

Since f has fixed point in $\mathbb{D}$, $|z_\varepsilon| \to 1$ as $\varepsilon \to 0$,

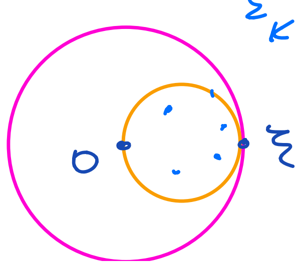


Consider

$$\partial B_{\rho_{hyp}}(z_\varepsilon, r_\varepsilon) =: N_\varepsilon$$

$$r_\varepsilon = \rho_{hyp}(z_\varepsilon, 0)$$

There is a subsequence (ε_k) s.t. $N_{\varepsilon_k} \to N_0$.



$f_\varepsilon(N_\varepsilon) \subset N_\varepsilon$ by Schwarz-Pick in limit

$$f(N_0) \subset N_0$$

$\Longrightarrow$ $f^n(0) \subset N_0$ $\forall n$

since $(f^n(0))$ has no accum. pts inside $\mathbb{D}$, then $f^n(0) \longrightarrow \xi$ as $n \to \infty$.